MODULAR LATTICE FOR C_o-OPERATORS.

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Abstract. We study modularity of the lattice $\mathtt{Lat}(T)$ of closed invariant subspaces for a C_0-operator T and find a condition such that $\mathtt{Lat}(T)$ is a modular. Furthermore, we provide a quasiaffinity preserving modularity.

Introduction

A partially ordered set is said to be a *lattice* if any two elements $\mathbf{M}$ and $\mathbf{N}$ of it have a least upper bound or supremum denoted by $\mathbf{M} \vee \mathbf{N}$ and a greatest lower bound or infimum denoted by $\mathbf{M} \cap \mathbf{N}$. For a Hilbert space H, $L(H)$ denotes the set of all bounded linear operators from H into H. For an operator T in $L(H)$, the set $\mathtt{Lat}(T)$ of all closed invariant subspaces for T is a lattice. For $\mathbf{L}$, $\mathbf{M}$, and $\mathbf{N}$ in $\mathtt{Lat}(T)$ such that $\mathbf{N} \subset \mathbf{L}$, if following identity is satisfied :

$$\mathbf{L} \cap (\mathbf{M} \vee \mathbf{N}) = (\mathbf{L} \cap \mathbf{M}) \vee \mathbf{N},$$

then $\mathtt{Lat}(T)$ is called *modular*. We study $\mathtt{Lat}(T)$ where T is a C_0-operator which were first studied in detail by B.Sz.-Nagy and C. Foias [4]. In this paper $\mathbf{D}$ denotes the open unit disk in the complex plane.

This paper is organized as follows. Section 1 contains preliminaries about operators of class C_0 and the Jordan model of C_0-operators.

For operators $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$, if $X \in \{A \in L(H) : AT_1 = T_2A\}$, then we define a function $X_* : \mathtt{Lat}(T_1) \to \mathtt{Lat}(T_2)$ as following:

$$X_*(M) = (XM)^-.$$

In Theorem 2.14, we provide a quasiaffinity Y such that Y_* preserves modularity. Furthermore, in section 2, we provide a definition and prove some fundamental results of *property (P)* which was introduced by H. Bercovici [2].

In Theorem 3.5, we prove that if $T \in L(H)$ is an operator of class C_0 with property (P), then $\mathtt{Lat}(T)$ is a modular lattice.

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1. C_0-Operators Relative to $\mathbf{D}$

1.1. **A Functional Calculus.** It is well-known that for every linear operator A on a finite dimensional vector space V over the field F, there is a minimal polynomial for A which is the (unique) monic generator of the ideal of polynomials over F which annihilate A. If the dimension of F is not finite, then generally there is no such a polynomial. However, to provide a function similar to a minimal polynomial, B. Sz.-Nagy and C. Foias focused on a contraction $T \in L(H)$ which is called to be *completely nonunitary*, i.e. there is no invariant subspace M for T such that the restriction $T|M$ of T to the space M is a unitary operator.

Let H be a subspace of a Hilbert space K and P_H be the orthogonal projection from K onto H. We recall that if $A \in L(K)$, and $T \in L(H)$, then A is said to be a *dilation* of T provided that for $n = 1, 2, ...,$

$$T^n = P_H A^n | H. \tag{1.1}$$

If A is an isometry (unitary operator) then A will be called an *isometric* (*unitary*) *dilation* of T. An isometric (unitary) dilation A of T is said to be *minimal* if no restriction of A to an invariant subspace is an isometric (unitary) dilation of T. B. Sz.-Nagy proved the following interesting result:

Proposition 1.1. [4] *Every contraction has a unitary dilation.*

Let $T \in L(H)$ be a completely nonunitary contraction with minimal unitary dilation $U \in L(K)$. For every polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ we have

$$p(T) = P_H p(U)|H, \tag{1.2}$$

and so this formula suggests that the functional calculus $p \to p(T)$ might be extended to more general functions p. Since the mapping $p \to p(T)$ is a homomorphism from the algebra of polynomials to the algebra of operators, we will extend it to a mapping which is also a homomorphism from an algebra to the algebra of operators. By Spectral Theorem, since $U \in L(H)$ is a normal operator, there is a unique *spectral measure* E on the Borel subsets of the spectrum of U denoted as usual by $\sigma(U)$ such that

$$U = \int_{\sigma(U)} z dE(z). \tag{1.3}$$

Since the spectral measure E of U is absolutely continuous with respect to Lebesgue measure on $\partial\mathbf{D}$, for $g \in L^\infty(\sigma(U), E)$, $g(U)$ can be defined as follows:

$$g(U) = \int_{\sigma(U)} g(z) dE(z). \tag{1.4}$$

It is clear that if g is a polynomial, then this definition agrees with the preceding one. Since the spectral measure of U is absolutely continuous with respect to Lebesgue measure on $\partial\mathbf{D}$, the expression $g(U)$ makes sense for every $g \in L^\infty = L^\infty(\partial\mathbf{D})$. We generalize formula (1.2), and so for $g \in L^\infty$, define $g(T)$ by

$$g(T) = P_H g(U)|H. \tag{1.5}$$

While the mapping $g \to g(T)$ is obviously linear, it is not generally multiplicative, i.e. it is not a homomorphism. Evidently it is convenient to find a subalgebra in L^∞ on which the functional calculus is multiplicative. Recall that H^∞ is the Banach

space of all (complex-valued) bounded analytic functions on the open unit disk $\mathbf{D}$ with supremum norm [4]. It turns out that H^∞ is the unique maximal algebra making the map a homomorphism between algebras. We know that H^∞ can be regarded as a subalgebra of $L^\infty(\partial\mathbf{D})$ [1].

We note that the functional calculus with H^∞ functions can be defined in terms of independent of the minimal unitary dilation. Indeed, if $u(z) = \sum_{n=0}^{\infty} a_n z^n$ is in H^∞, then

$$u(T) = \lim_{r\to 1} u(rT) = \lim_{r\to 1} \sum_{n=0}^{\infty} a_n r^n T^n, \tag{1.6}$$

where the limit exists in the strong operator topology.

B. Sz.-Nagy and C. Foias introduced this important functional calculus for completely nonunitary contractions.

Proposition 1.2. *Let $T \in L(H)$ be a completely nonunitary contraction. Then there is a unique algebra representation Φ_T from H^∞ into $L(H)$ such that :*

(i) $\Phi_T(1) = I_H$, where $I_H \in L(H)$ is the identity operator;
(ii) $\Phi_T(g) = T$, if $g(z) = z$ for all $z \in \boldsymbol{D}$;
(iii) Φ_T is continuous when H^∞ and $L(H)$ are given the weak*-topology.
(iv) Φ_T is contractive, i.e. $\|\Phi_T(u)\| \leq \|u\|$ for all $u \in H^\infty$.

We simply denote by $u(T)$ the operator $\Phi_T(u)$.

B.Sz.- Nagy and C. Foias [4] defined the *class* C_0 relative to the open unit disk $\mathbf{D}$ consisting of completely nonunitary contractions T on H such that the kernel of Φ_T is not trivial. If $T \in L(H)$ is an operator of class C_0, then

$$\ker\, \Phi_T = \{u \in H^\infty : u(T) = 0\}$$

is a weak*-closed ideal of H^∞, and hence there is an inner function generating ker Φ_T. The *minimal function* m_T of an operator of class C_0 is the generator of ker Φ_T, and it seems as a substitute for the minimal polynomial. Also, m_T is uniquely determined up to a constant scalar factor of absolute value one [1]. The theory of class C_0 relative to the open unit disk has been developed by B.Sz.- Nagy, C. Foias ([4]) and H. Bercovici ([1]).

1.2. **Jordan Operator.** We know that every $n \times n$ matrix over an algebraically closed field F is similar to a unique Jordan canonical form. To extend that theory to the C_0 operator $T \in L(H)$, B.Sz.- Nagy and C. Foias [4] introduced a weaker notion of equivalence. They defined a *quasiaffine transform* of T which is bounded operator T' defined on a Hilbert space H' such that there exists an injective operator $X \in L(H, H')$ with dense range in H' satisfying $T'X = XT$. We write

$$T \prec T'$$

if T is a quasiaffine transform of T'. Instead of similarity, they introduced *quasisimilarity* of two operators, namely, T and T' are *quasisimilar*, denoted by

$$T \sim T',$$

if $T \prec T'$ and $T' \prec T$.

Given an inner function $\theta \in H^\infty$, the *Jordan block* $S(\theta)$ is the operator acting on $H(\theta) = H^2 \ominus \theta H^2$, which means the orthogonal complement of θH^2 in the Hardy space H^2, as follows :

$$S(\theta) = P_{H(\theta)} S | H(\theta) \tag{1.7}$$

where $S \in L(H^2)$ is the unilateral shift operator defined by

$$(Sf)(z) = zf(z)$$

and $P_{H(\theta)} \in L(H^2)$ denotes the orthogonal projection of H^2 onto $H(\theta)$.

Proposition 1.3. [1] *For every inner function θ in H^∞, the operator $S(\theta)$ is of class C_0 and its minimal function is θ.*

Let θ and θ' be two inner functions in H^∞. We say that θ *divides* θ' (or $\theta|\theta'$) if θ' can be written as $\theta' = \theta\cdot\phi$ for some $\phi \in H^\infty$. It is clear that $\phi \in H^\infty$ is also inner. We will use the notation $\theta \equiv \theta'$ if $\theta|\theta'$ and $\theta'|\theta$.

Proposition 1.4. [1] *Let $T_1 \in L(H)$ and $T_2 \in L(H)$ be two completely nonunitary contarctions of class C_0. If T_1 and T_2 are quasisimilar, then $m_{T_1} \equiv m_{T_2}$.*

From Proposition 1.3 and Proposition 1.4, we can easily see that for every inner functions θ_1 and θ_2 in H^∞, if $S(\theta_1)$ and $S(\theta_2)$ are quasisimilar, then $\theta_1 \equiv \theta_2$. Conversely,

Proposition 1.5. [1] *Let θ_1 and θ_2 be inner functions in H^∞. If $\theta_1 \equiv \theta_2$, then $S(\theta_1)$ and $S(\theta_2)$ are quasisimilar.*

Let γ be a cardinal number and

$$\Theta = \{\theta_\alpha \in H^\infty : \alpha < \gamma\}$$

be a family of inner functions. Then Θ is called a *model function* if $\theta_\alpha|\theta_\beta$ whenever $\text{card}(\beta) \leq \text{card}(\alpha) < \gamma$. The *Jordan operator* $S(\Theta)$ determined by the model function Θ is the C_0 operator defined as

$$S(\Theta) = \bigoplus_{\alpha<\gamma'} S(\theta_\alpha)$$

where $\gamma' = \min\{\beta : \theta_\beta \equiv 1\}$.

We will call $S(\Theta)$ the *Jordan model* of the operator T if

$$S(\Theta) \sim T,$$

and in the sequel $\bigoplus_{i<\gamma'} S(\theta_i)$ always means a *Jordan operator* determined by a model function.

By using Jordan blocks, C_0-operators relative to the open unit disk $\mathbf{D}$ can be classified ([1] Theorem 5.1) :

Theorem 1.6. *Any C_0-operator T relative to the open unit disk $\boldsymbol{D}$ acting on a Hilbert space is quasisimilar to a unique Jordan operator.*

Theorem 1.7. *If Θ and Θ' are two model functions and $S(\Theta) \prec S(\Theta')$, then $\Theta \equiv \Theta'$ and hence $S(\Theta) = S(\Theta')$.*

From Theorem 1.6 and Theorem 1.7, we can conclude that " $\prec$ " is an equivalence relation on the set of C_0-operators.

2. Lattice of subspaces

2.1. **Modular Lattice.** Let H be a Hilbert space. If $F_i (i \in I)$ is a subset of H, then the closed linear span of $\bigcup_i F_i$ will be denoted by $\bigvee_i F_i$. The collection of all subspaces of a Hilbert space is a *lattice*. This means that the collection is partially ordered (by inclusion), and that any two elements $\mathbf{M}$ and $\mathbf{N}$ of it have a least

upper bound or supremum (namely the span $\mathbf{M} \vee \mathbf{N}$) and a greatest lower bound or infimum (namely the intersection $\mathbf{M} \cap \mathbf{N}$). A lattice is called *distributive* if

$$\mathbf{L} \cap (\mathbf{M} \vee \mathbf{N}) = (\mathbf{L} \cap \mathbf{M}) \vee (\mathbf{L} \cap \mathbf{N}) \tag{2.1}$$

for any element $\mathbf{L}$, $\mathbf{M}$, and $\mathbf{N}$ in the lattice.

In the equation (2.1), if $\mathbf{N} \subset \mathbf{L}$, then $\mathbf{L} \cap \mathbf{N} = \mathbf{N}$, and so the identity becomes

$$\mathbf{L} \cap (\mathbf{M} \vee \mathbf{N}) = (\mathbf{L} \cap \mathbf{M}) \vee \mathbf{N} \tag{2.2}$$

If the identity (2.2) is satisfied whenever $\mathbf{N} \subset \mathbf{L}$, then the lattice is called *modular.*

For an arbitrary operator $T \in L(H)$, $\mathtt{Lat}(T)$ denotes the collection of all closed invariant subspaces for T. The following fact is well-known [3].

Proposition 2.1. *The lattice of subspaces of a Hilbert space H is modular if and only if* dim *H is finite.*

We will think about $\mathtt{Lat}(T)$ for a C_0-operator T.

Definition 2.2. The *cyclic multiplicity* μ_T of an operator $T \in L(H)$ is the smallest cardinal of a subset $A \subset H$ with the property that $\bigvee_{n=0}^{\infty} T^n A = H$. The operator T is said to be *multiplicity-free* if $\mu_T = 1$.

Thus μ_T is the smallest number of cyclic subspaces for T that are needed to generate H, and T is multiplicity-free if and only if it has a cyclic vector.

2.2. **Property** (P)**.** Let H be a Hilbert space and for an operator $T \in L(H)$, T^* denote the adjoint of T. It is well known that H is finite-dimensional if and only if every operator $X \in L(H)$, with the property $\ker(X) = \{0\}$, also satisfies $\ker(X^*) = \{0\}$. The following definition is a natural extension of finite dimensionality.

Definition 2.3. An operator $T \in L(H)$ is said to have *property (P)* if every operator $X \in \{T\}'$ with the property that $\ker(X) = \{0\}$ is a quasiaffinity, i.e., $\ker(X^*) = \ker(X) = \{0\}$.

From the fact that the commutant $\{0\}'$ of zero operator on H coincides with $L(H)$, we can see that H is finite-dimensional if and only if the zero operator on H has property (P).

Let T_1 and T_2 be operators in $L(H)$. Suppose that

$$X \in \{A \in L(H) : AT_1 = T_2 A\}.$$

If M is in $\mathtt{Lat}(T_1)$, then $(XM)^-$ is in $\mathtt{Lat}(T_2)$. By using these facts, we define a function X_* from $\mathtt{Lat}(T_1)$ to $\mathtt{Lat}(T_2)$ as following :

$$X_*(M) = (XM)^-. \tag{2.3}$$

The operator X is said to be a (T_1, T_2)-*lattice-isomorphism* if X_* is a bijection of $\mathtt{Lat}(T_1)$ onto $\mathtt{Lat}(T_2)$. We will use the name lattice-isomorphism instead of (T_1, T_2)-lattice-isomorphism if no confusion may arise.

If $X \in \{A \in L(H) : AT_1 = T_2 A\}$, then $X^* T_2^* = T_1^* X^*$. Thus $(X^*)_* : \mathtt{Lat}(T_2^*) \to \mathtt{Lat}(T_1^*)$ is well-defined by

$$(X^*)_*(M') = (X^* M')^-.$$

Proposition 2.4. [1] *(`Theorem 7.1.9`) Suppose that $T \in L(H)$ is an operator of class C_0 with Jordan model $\bigoplus_\alpha S(\theta_\alpha)$. Then T has property (P) if and only if*

$$\bigwedge_{j<\omega} \theta_j \equiv 1.$$

Thus, if T has property (P), then H is separable and T^* also has property (P).

Proposition 2.5. [1] *An operator T of class C_0 fails to have property (P) if and only if T is quasisimilar to $T|N$, where N is a proper invariant subspace for T.*

Proposition 2.6. [1]*(Lemma 7.1.20) Assume that $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$ are two operators, and $X \in \{A \in L(H_1, H_2) : AT_1 = T_2A\}$. If the mapping X_* is onto $\mathtt{Lat}(T_2)$ if and only if $(X^*)_*$ is one-to-one on $\mathtt{Lat}(T_2^*)$.*

Corollary 2.7. *Assume that $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$ are two operators, and $X \in \{A \in L(H_1, H_2) : AT_1 = T_2A\}$. The mapping X_* is one-to-one on $\mathtt{Lat}(T_1)$ if and only if $(X^*)_*$ is onto $\mathtt{Lat}(T_1^*)$.*

Proof. Since $XT_1 = T_2X$, $T_1^*X^* = X^*T_2^*$. By Proposition 2.6, $(X^*)_*$ is onto $\mathtt{Lat}(T_1^*)$ if and only if $(X^{**})_* = X_*$ is one-to-one on $\mathtt{Lat}(T_1)$. □

From Proposition 2.6 and Corollary 2.7, we obtain the following result.

Corollary 2.8. *If $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$ are two operators, and $X \in \{A \in L(H_1, H_2) : AT_1 = T_2A\}$, then X is a lattice-isomorphism if and only if X^* is a lattice-isomorphism.*

Proposition 2.9. [1] *(Proposition 7.1.21) Assume that $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$ are two quasisimilar operators of class C_0, and $X \in \{A \in L(H_1, H_2) : AT_1 = T_2A\}$ is an injection. If T_1 has property (P), then X is a lattice-isomorphism.*

Recall that if T is an operator on a Hilbert space, then ker T = (ran $T^*)^\perp$ and ker T^* = (ran $T)^\perp$.

Corollary 2.10. *Assume that $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$ are two quasisimilar operators of class C_0, and $X \in \{A \in L(H_1, H_2) : AT_1 = T_2A\}$ has dense range. If T_2 has property (P), then X is a lattice-isomorphism.*

Proof. Since $XT_1 = T_2X$, $T_1^*X^* = X^*T_2^*$. Let $Y = X^*$ and so

$$YT_2^* = T_1^*Y. \tag{2.4}$$

From the fact that $\ker Y = \ker(X^*) = (\mathtt{ran}\ X)^\perp = \{0\}$, we conclude that Y is injective. Since T_2 has property (P), so does T_2^* by Proposition 2.4. By Proposition 2.9 and equation (2.4), $Y = X^*$ ia a lattice-isomorphism. From Corollary 2.8, it is proven that X is a lattice-isomorphism. □

Corollary 2.11. *Suppose that $T_i \in L(H_i)(i = 1, 2)$ is a C_0-operator and T_1 has property (P). If $X \in \{A \in L(H_1, H_2) : AT_1 = T_2A\}$ and X is an injection, then X is a lattice-isomorphism.*

Proof. Define $Y : H_1 \to (XH_1)^-$ by

$$Yh = Xh \text{ for any } h \in H_1.$$

Since X is an injection, so is Y. Clearly, Y has dense range. Note that $(XH_1)^-$ is invariant for T_2. By definition of Y,

$$YT_1 = (T_2|(XH_1)^-)Y. \tag{2.5}$$

It follows that $T_1 \prec (T_2|(XH_1)^-)$ and so $T_1 \sim (T_2|(XH_1)^-)$. By Proposition 2.9, it is proven. □

Corollary 2.12. *Suppose that $T_i \in L(H_i)(i = 1, 2)$ is a C_0-operator and T_2 has property (P). If $X \in \{A \in L(H_1, H_2) : AT_1 = T_2A\}$ and X has a dense range, then X is a lattice-isomorphism.*

Proof. By assumption, $X^*T_2^* = T_1^*X^*$. Since T_2 has property (P), by Proposition 2.4, so does T_2^*.

Because X has dense range, $X^* : H_2 \to H_1$ is an injection. By Corollary 2.11, X^* is a lattice isomorphism. From Corollary 2.8, X is also a lattice isomorphism. $\square$

2.3. **Quasi-Affinity and Modular Lattice.** For operators $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$, if $Y \in \{B \in L(H_1, H_2) : BT_1 = T_2B\}$, then we define a function

$$Y_* : \mathtt{Lat}(T_1) \to \mathtt{Lat}(T_2)$$

the same way as equation (2.3). For any $N \in \mathtt{Lat}(T_2)$, if $M = Y^{-1}(N)$, then $YT_1(M) = T_2Y(M) \subset T_2N \subset N$ and so $T_1(M) \subset M$. It follows that

$$M = Y^{-1}(N) \in \mathtt{Lat}(T_1)$$

for any $N \in \mathtt{Lat}(T_2)$. If Y is invertible, that is, T_1 and T_2 are similar, and $\mathtt{Lat}(T_1)$ is modular, then clearly, $\mathtt{Lat}(T_2)$ is also modular. In this section, we consider when T_1 and T_2 are quasi-similar instead of similar, and find an assumption in Theorem 2.14 such that $\mathtt{Lat}(T_2)$ is modular, whenever $\mathtt{Lat}(T_1)$ is modular.

Proposition 2.13. *Let $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$. Suppose that $Y \in \{B \in L(H_1, H_2) : BT_1 = T_2B\}$ and for any $N \in \mathtt{Lat}(T_2)$, the condition $M = Y^{-1}(N)$ implies that $Y_*(M) = N$.*

Then for any $M_i = Y^{-1}(N_i)$ with $N_i \in \mathtt{Lat}(T_2)$ $(i = 1, 2)$,

$$Y_*(M_1 \cap M_2) = Y_*(M_1) \cap Y_*(M_2).$$

Proof. Assume that $N_i \in \mathtt{Lat}(T_2)$ and $M_i = Y^{-1}(N_i)$ for $i = 1, 2$. Then by assumption, we obtain

$$Y_*(M_i) = N_i. \tag{2.6}$$

Since $Y^{-1}(N_1 \cap N_2) = Y^{-1}(N_1) \cap Y^{-1}(N_2) = M_1 \cap M_2$, by assumption,

$$Y_*(M_1 \cap M_2) = N_1 \cap N_2$$

which proves that $Y_*(M_1 \cap M_2) = Y_*(M_1) \cap Y_*(M_2)$ by equation (2.6).

$\square$

Theorem 2.14. *Let $T_1 \in L(H_1)$ be a quasiaffine transform of $T_2 \in L(H_2)$ and $Y \in \{B \in L(H_1, H_2) : BT_1 = T_2B\}$ be a quasiaffinity.*

If $Y_* : \mathtt{Lat}(T_1) \to \mathtt{Lat}(T_2)$ is onto and $\mathtt{Lat}(T_1)$ is modular, then $\mathtt{Lat}(T_2)$ is also modular.

Proof. Suppose that $\mathtt{Lat}(T_2)$ is not modular. Then there are invariant subspaces $N_i (i = 1, 2, 3)$ for T_2 such that

$$N_3 \subset N_1, \tag{2.7}$$

and

$$(N_1 \cap N_2) \vee N_3 \neq N_1 \cap (N_2 \vee N_3).$$

Let

$$M_i = Y^{-1}(N_i), \tag{2.8}$$

for $i = 1, 2, 3$. Since $YT_1 = T_2Y$, definition (2.8) of M_i implies that for $i = 1, 2, 3$,

$$YT_1(M_i) = T_2Y(M_i) \subset T_2N_i \subset N_i.$$

It follows that $T_1M_i \subset Y^{-1}(N_i) = M_i$ for $i = 1, 2, 3$. Thus M_i is a closed invariant subspace for T_1. Condition (2.7) implies that

$$M_3 \subset M_1.$$

Since $Y(M_i) \subset N_i$, for $i = 1, 2, 3$,

$$Y_*(M_i) = (Y(M_i))^- \subset N_i. \tag{2.9}$$

Since Y_* is onto, there is a function $\phi : \mathtt{Lat}(T_2) \to \mathtt{Lat}(T_1)$ such that $Y_* \circ \phi$ is the identity mapping on $\mathtt{Lat}(T_2)$. Hence for $i = 1, 2, 3$,

$$Y_*(\phi(N_i)) = Y(\phi(N_i))^- = N_i.$$

It follows that for $i = 1, 2, 3$,

$$\phi(N_i) \subset M_i. \tag{2.10}$$

Since $Y_* \circ \phi$ is the identity mapping on $\mathtt{Lat}(T_2)$, (2.10) implies that for $i = 1, 2, 3$,

$$N_i = Y_*(\phi(N_i)) \subset Y_*(M_i). \tag{2.11}$$

By (2.9) and (2.11), we get

$$Y_*(M_i) = N_i, \tag{2.12}$$

for $i = 1, 2, 3$. Hence we can easily see that function Y satisfies the assumptions of Proposition 2.13.

Thus by Proposition 2.13 and equation (2.12),

$$Y_*[M_1 \cap (M_2 \vee M_3)] = Y_*(M_1) \cap Y_*(M_2 \vee M_3) = N_1 \cap (N_2 \vee N_3). \tag{2.13}$$

Since $M_1 \cap M_2 = Y^{-1}(N_1) \cap Y^{-1}(N_2) = Y^{-1}(N_1 \cap N_2)$, by the same way as above, we obtain

$$Y_*(M_1 \cap M_2) = N_1 \cap N_2. \tag{2.14}$$

By equations (2.12) and (2.14), we obtain

$$Y_*[(M_1 \cap M_2) \vee M_3] = (N_1 \cap N_2) \vee N_3. \tag{2.15}$$

Since $(N_1 \cap N_2) \vee N_3 \neq N_1 \cap (N_2 \vee N_3)$, from equations (2.13) and (2.15), we can conclude that

$$(M_1 \cap M_2) \vee M_3 \neq M_1 \cap (M_2 \vee M_3).$$

Therefore $\mathtt{Lat}(T_1)$ is not modular.

□

3. Modular Lattice for C_0-Operators with Property (P)

We provide some operators, say T, of class C_0 such that $\mathtt{Lat}(T)$ is modular.

Proposition 3.1. [1] *Let θ be a nonconstant inner function in H^∞. Then every invariant subspace M of $S(\theta)$ has the form*

$$\phi H^2 \ominus \theta H^2$$

for some inner devisor ϕ of θ.

We can easily check that if $\mathbf{M}_1 = \theta_1 H^2 \ominus \theta H^2$ and $\mathbf{M}_2 = \theta_2 H^2 \ominus \theta H^2$ where θ_i $(i = 1, 2)$ is an inner inner devisor of θ, then

$$\mathbf{M}_1 \cap \mathbf{M}_2 = (\theta_1 \vee \theta_2) H^2 \ominus \theta H^2 \tag{3.1}$$

and

$$\mathbf{M}_1 \vee \mathbf{M}_2 = (\theta_1 \wedge \theta_2) H^2 \ominus \theta H^2 \tag{3.2}$$

where $\theta_1 \wedge \theta_2$ and $\theta_1 \vee \theta_2$ denote the greatest common inner divisor and least common inner multiple of θ_1 and θ_2, respectively. Note that if $\mathbf{M}_1 \subset \mathbf{M}_2$, then

$$\theta_2 | \theta_1. \tag{3.3}$$

Lemma 3.2. *If θ is an inner function in H^∞, then $\mathtt{Lat}(S(\theta))$ is distributive.*

Proof. Let $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{M}_3$ be invariant subspaces for $S(\theta)$. Then by Proposition 3.1, there are nonconstant inner functions θ_1, θ_2, and θ_3 in H^∞ such that

$$\mathbf{M}_i = \theta_i H^2 \ominus \theta H^2 \text{ for } i = 1, 2, 3.$$

From equations (3.1) and (3.2), we obtain that

$$\mathbf{M}_1 \cap (\mathbf{M}_2 \vee \mathbf{M}_3) = (\theta_1 \vee (\theta_2 \wedge \theta_3)) H^2 \ominus \theta H^2, \tag{3.4}$$

and

$$(\mathbf{M}_1 \cap \mathbf{M}_2) \vee (\mathbf{M}_1 \cap \mathbf{M}_3) = ((\theta_1 \vee \theta_2) \wedge (\theta_1 \vee \theta_3) H^2 \ominus \theta H^2. \tag{3.5}$$

Since $\theta_1 \vee (\theta_2 \wedge \theta_3) = (\theta_1 \vee \theta_2) \wedge (\theta_1 \vee \theta_3)$, by equations (3.4) and (3.5), this lemma is proven.

□

In this section, we will consider a sufficient condition for $\mathtt{Lat}(T)$ of a C_0-operator T to be modular.

Proposition 3.3. [1] *(Proposition 2.4.3) Let $T \in L(H)$ be a completely nonunitary contraction, and M be an invariant subspace for T. If*

$$T = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix} \tag{3.6}$$

is the triangularization of T with respect to the decomposition $H = M \oplus (H \ominus M)$, then T is of class C_0 if and only if T_1 and T_2 are operators of class C_0.

Proposition 3.4. [1] *(Corollary 7.1.17) Let $T \in L(H)$ is an operator of class C_0, M be an invariant subspace for T, and*

$$T = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix} \tag{3.7}$$

be the triangularization of T with respect to the decomposition $H = M \oplus (H \ominus M)$. Then T has property (P) if and only if T_1 and T_2 have property (P).

Let H and K be Hilbert spaces and $H \oplus K$ denote the algebraic direct sum. Recall that $H \oplus K$ is also a Hilbert space with an inner product

$$(\langle h_1, k_1\rangle, \langle h_2, k_2\rangle) = (h_1, h_2) + (k_1, k_2)$$

Theorem 3.5. *Let $T \in L(H)$ be an operator of class C_0 with property (P). Then $\mathtt{Lat}(T)$ is a modular lattice.*

Proof. Suppose that T has property (P) and let M_i $(i = 1, 2, 3)$ be an invariant subspace for T such that $M_3 \subset M_1$. Then evidently,

$$(M_1 \cap M_2) \vee M_3 \subset M_1 \cap (M_2 \vee M_3). \tag{3.8}$$

Let $T_i = T|M_i$ $(i = 1, 2, 3)$. Define a linear transformation $X : M_2 \oplus M_3 \to M_2 \vee M_3$ by

$$X(a_2 \oplus a_3) = a_2 + a_3$$

for $a_2 \in M_2$ and $a_3 \in M_3$.

Then for $a_2 \oplus a_3 \in M_2 \oplus M_3$ with $\|a_2 \oplus a_3\| \le 1$, $\|X(a_2 \oplus a_3)\| = \|a_2 + a_3\| \le \|a_2\| + \|a_3\| \le 2$. It follows that $\|X\| \le 2$ and so X is bounded.

Since $M_2 \vee M_3$ is generated by $\{a_2 + a_3 : a_2 \in M_2 \text{ and } a_3 \in M_3\}$, X has dense range. By definition of T_i $(i = 1, 2, 3)$,

$$X(T_2 \oplus T_3)(a_2 \oplus a_3) = Ta_2 + Ta_3$$

and

$$(T|M_2 \vee M_3)X(a_2 \oplus a_3) = Ta_2 + Ta_3.$$

Thus

$$X(T_2 \oplus T_3) = (T|M_2 \vee M_3)X.$$

By Proposition 3.3, $T_2 \oplus T_3$ and $T|M_2 \vee M_3$ are of class C_0 and since T has property (P), by Proposition 3.4, we conclude that $T|M_2 \vee M_3$ also has Property (P). By Corollary 2.12, X is a lattice-isomorphism.

Thus $X_* : \mathtt{Lat}(T_2 \oplus T_3) \to \mathtt{Lat}(T|M_2 \vee M_3)$ is onto. Let

$$M = \{a_2 \oplus a_3 \in M_2 \oplus M_3 : a_2 + a_3 \in M_1\}. \tag{3.9}$$

Since $M = X^{-1}(M_1)$, M is a closed subspace of $M_2 \oplus M_3$. Evidently, M is invariant for $T_2 \oplus T_3$. From the equation (3.9), we conclude that

$$M = (M_1 \cap M_2) \oplus M_3. \tag{3.10}$$

Since $X^{-1}(M_1 \cap (M_2 \vee M_3)) = \{a_2 \oplus a_3 \in M_2 \oplus M_3 : a_2 + a_3 \in M_1 \cap (M_2 \vee M_3)\} = \{a_2 \oplus a_3 \in M_2 \oplus M_3 : a_2 + a_3 \in M_1\}$,

$$X^{-1}(M_1 \cap (M_2 \vee M_3)) = M$$

Since X is a lattice-isomorphism,

$$X_* M = (XM)^- = M_1 \cap (M_2 \vee M_3). \tag{3.11}$$

By equation (3.10) and definition of X,

$$X_* M = (XM)^- \subset (M_1 \cap M_2) \vee M_3. \tag{3.12}$$

From (3.11) and (3.12), we conclude that

$$M_1 \cap (M_2 \vee M_3) \subset (M_1 \cap M_2) \vee M_3. \tag{3.13}$$

Thus if T has property (P), then by (3.8) and (3.13), we obtain that

$$M_1 \cap (M_2 \vee M_3) = (M_1 \cap M_2) \vee M_3.$$

$\square$

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